Invariants of graded modules via relative Gröbner bases

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Universität Bremen, 2 March 2026

Table of Contents

Open problems on Hilbert series

Given homogeneous ideal $I \subseteq R = K[x_1, \ldots, x_n]$ and homogeneous element $h \in R \setminus I$ of degree d, is it true that if the generators of I and h are generic of prescribed degrees, then all the multiplication maps $\cdot h : (R/I)_e \to (R/I)_{e+d}$ must have maximal rank?

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Fröberg's conjecture (1985)

The Hilbert series for the quotient of R by m general forms of degrees $a_1, \ldots, a_m$ is

$$\left[\frac{\prod_{i=1}^{m}(1 - t^{a_i})}{(1 - t)^n}\right],$$

where $[\ldots]$ denotes truncation at the first non-positive coefficient.

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Overarching goal

Understand graded ideals and algebras in generic position or defined by generic forms, using constructive methods of Computer Algebra.

Structural information and corresponding methods

Structure	Method
Hilbert series	

Structural information and corresponding methods

Structure	Method
• Hilbert series	• Gröbner bases

Structural information and corresponding methods

Structure

- Hilbert series
- Combinatorial decomposition

Method

- Gröbner bases

Structural information and corresponding methods

Structure	Method
• Hilbert series	• Gröbner bases
• Combinatorial decomposition	• Janet-like bases

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Structure	Method
Hilbert series	Gröbner bases
Combinatorial decomposition	Janet-like bases
Generic structure	Pommaret-like bases
Resolutions and Betti numbers	Module Gröbner bases
Invariants over quotient rings	Relative Pommaret-like bases

Acknowledgements

This talk is partly based on joint work with:

- Amir Hashemi and Werner Seiler (relative Gröbner and involutive-like bases and invariants),
- Fritz Grimpen and Anastasios Stefanou (multigraded relative module Gröbner bases),
- Filip Jonsson Kling, Samuel Lundqvist, Fatemeh Mohammadi, and Eduardo Sáenz-de-Cabezón (principal ideals in AMCIs).

Table of Contents

Polynomials and Ideals

- $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_n]$: polynomial ring
- $\mathcal{T}$: set of terms $\mathbf{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$.
- Polynomial: Finite sum of monomials $c\mathbf{x}^\mu$ ($c \in \mathbb{K}$).
- $\deg(\mathbf{x}^\mu) = \sum_{i=1}^n \mu_i$: degree of the term $\mathbf{x}^\mu$
- *Generating set* or *basis* of ideal $\mathcal{I} \trianglelefteq \mathcal{R}$: $G = \{g_1, \ldots, g_m\} \subseteq \mathcal{I}$ with

$$\langle G \rangle := \left\{ \sum h_i g_i \mid h_1, \ldots, h_m \in \mathcal{R} \right\} = \mathcal{I}$$

- *Homogeneous polynomial of degree d*: $f = \sum c_i t_i$ with $\deg(t_i) = d$.
- *Homogeneous ideal*: Ideal generated by homogeneous polynomials
- If $G = \{t_1, \ldots, t_m\} \subset \mathcal{T} \rightsquigarrow \langle G \rangle$ *monomial ideal*.

We work with homogeneous and monomial ideals.

Term Orderings

- $\prec$: fixed term ordering $[\forall s, t, u \in \mathcal{T}\colon 1 \prec t$ and $(s \prec t) \Rightarrow (su \prec tu)]$
- $\mathrm{lt}(f)$: leading term of $0 \neq f \in \mathcal{R}$
- $\mathrm{lt}(\mathcal{I}) = \langle \mathrm{lt}(f) \mid f \in \mathcal{I} \rangle$: leading ideal of $0 \neq \mathcal{I} \trianglelefteq \mathcal{R}$
- Via term orderings, can use monomial ideals to study general ideals.

Gröbner Bases

Definition

$G = \{g_1, \ldots, g_m\} \subset \mathcal{I}$ Gröbner basis of ideal $\mathcal{I} \trianglelefteq \mathcal{R}$, if $\langle \mathrm{lt}(g_1), \ldots, \mathrm{lt}(g_m) \rangle = \mathrm{lt}(\mathcal{I})$.

Definition

Let $f, g \in \mathcal{R}$ be two monic polynomials. Their S-polynomial is $S(f,g) = (\mathsf{lcm}(\mathrm{lt}(f), \mathrm{lt}(g))/\mathrm{lt}(f)) \cdot f - (\mathsf{lcm}(\mathrm{lt}(f), \mathrm{lt}(g))/\mathrm{lt}(g)) \cdot g$.

Theorem (Buchberger 1965)

One can compute a Gröbner basis of any given ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ by producing new leading terms via S-polynomials.

Remark

For a fixed term order, the reduced Gröbner basis (autoreduced set of monic polynomials whose leading terms minimally generate the leading ideal) is unique.

Hilbert series

Definition

Let $M = \bigoplus_{d \geq 0} M_d$ be a finitely generated graded R-module with degree-d components M_d. The *Hilbert series* of M is

$$\mathrm{HS}(M; t) = \sum_{d=0}^{\infty} \dim_{\mathbf{k}}(M_d)\, t^d \,.$$

Example

For $A = \mathbf{k}[x_1, x_2]/(x_1^2, x_2^3)$, we have $\mathrm{HS}(A; t) = 1 + 2t + 2t^2 + t^3$.

Remark

Let $\{0\} \neq I \subseteq R$ be a homogeneous ideal, and $\prec$ a monomial ordering compatible with degrees. Then, $\mathrm{HS}(R/I; t) = \mathrm{HS}(R/\operatorname{in}_{\prec}(I); t)$.

Involutive Divisions and Bases

Involutive division L: Restricted division on $\mathcal{T}$ (Gerdt–Blinkov 1998).

- Given set $H \subset \mathcal{T}$ and $\mathbf{x}^\mu \in H$, assigns multiplicative variables $L(\mathbf{x}^\mu, H) \subseteq \{x_1, \dots, x_n\}$.
- The involutive cones $\mathcal{C}_L(\mathbf{x}^\mu, H) = \{\mathbf{x}^\nu \mid \mathbf{x}^\nu/\mathbf{x}^\mu \in \mathbb{K}[L(\mathbf{x}^\mu, H)]\}$ intersect only trivially as $\mathbf{x}^\mu$ varies in H.
- Filter axiom: If $H \supseteq H'$, then $L(\mathbf{x}^\mu, H') \supseteq L(\mathbf{x}^\mu, H)$.
- Polynomial set $H \subset \mathcal{I} \trianglelefteq \mathcal{R}$ *L-involutive basis* of ideal $\mathcal{I}$, if

$$\bigsqcup_{\mathbf{x}^\mu \in \operatorname{lt}(H)} \mathcal{C}_L(\mathbf{x}^\mu, \operatorname{lt}(H)) = \operatorname{lt}(\mathcal{I}).$$

Every involutive basis is a Gröbner basis.

Janet and Janet-like Divisions

Example (Janet division)

x_i is Janet mult. for $t = x_1^{d_1} \cdots x_n^{d_n} \in U$ if
$d_i = \max\{\deg_{x_i}(u) \mid u \in U,\ \deg_j(u) = d_j,\ i+1 \leq j \leq n\}$.

Example (Gerdt and Blinkov, 2005)

Janet-like division: Non-multiplicative powers instead of variables

Example

$U = \{x_1^{10}, x_2^{10}\} \rightsquigarrow$ Set of Janet-like non-multiplicative powers for x_1^{10} is $\{x_2^{10}\}$ however set of non-multiplicative variables $\{x_2\}$.

This helps to construct smaller bases!

Let $\mathcal{I} = \langle x_1^{10}, x_2^{10} \rangle$, then $\{x_1^{10}, x_1^{10}x_2, \ldots, x_1^{10}x_2^9, x_2^{10}\}$ is the Janet basis (there is a unique minimal JB), however $\{x_1^{10}, x_2^{10}\}$ is the Janet-like basis.

Janet-like Tree

Given $U = \{t_1, \ldots, t_m\} \subset \mathcal{T}$, the Janet-like tree of U contains nodes with labels (t, V, M), where...

1. ... $t \in \mathcal{T}$ encodes a Janet class of U,
2. ... V is the set of variables multiplicative for each term in this class.

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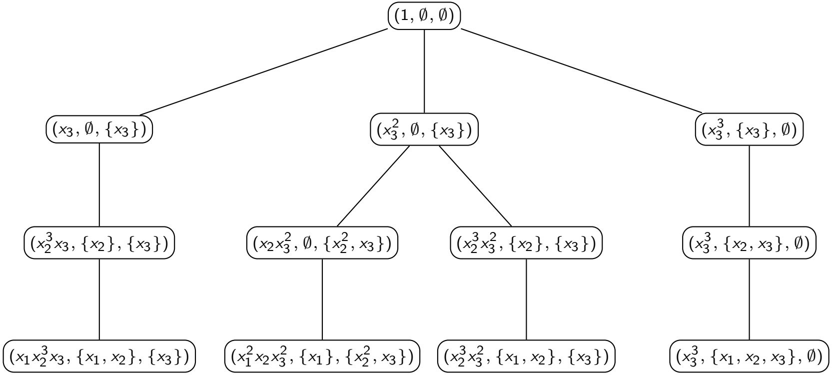
1. ... $t \in \mathcal{T}$ encodes a Janet class of U,
2. ... V is the set of variables multiplicative for each term in this class.
3. ... M is the set of non-multiplicative powers common to all terms in this class.

Application

The Janet-like tree of a Janet-like basis of $\mathcal{I}$ encodes a cone decomposition of $\mathcal{R}/\mathcal{I}$ and the irreducible decomposition of $\mathcal{I}$, and thus also the Hilbert series of $\mathcal{I}$ (Hashemi et al. 2022). It also gives the *standard pairs* (Sturmfels et al. 1995) of the quotient (Hashemi et al. 2025+).

Janet-like Tree of Janet-like Basis



Janet-like tree of Janet-like basis

$$B = \{x_1 x_2^3 x_3,\ x_1^2 x_2 x_3^2,\ x_2^3 x_3^2,\ x_3^3\} \subset \mathbb{K}[x_1, x_2, x_3].$$

Example Complementary Decomposition

Example

Complementary decomposition for $\mathcal{I} = \langle x_1 x_2^3 x_3,\ x_1^2 x_2 x_3^2,\ x_2^3 x_3^2,\ x_3^3 \rangle$.
Cones are given as pairs of vertex and set of multiplicative variables.

$$
\begin{aligned}
&(1, \{x_1, x_2\}),\\
&(x_3, \{x_1\}), (x_2 x_3, \{x_1\}), (x_2^2 x_3, \{x_1\}),\\
&(x_2^3 x_3, \{x_2\}),\\
&(x_3^2, \{x_1\}),\\
&(x_2 x_3^2, \emptyset), (x_2^2 x_3^2, \emptyset), (x_1 x_2 x_3^2, \emptyset), (x_1 x_2^2 x_3^2, \emptyset).
\end{aligned}
$$

The cones in each line are encoded in a single node of the Janet-like tree.

Pommaret Bases and Quasi-stable Ideals

Definition

Pommaret division P: term $\mathbf{x}^\mu$ with $\mu = (0, \ldots, 0, \mu_k, \ldots, \mu_n)$ and $\mu_k > 0$ $\rightsquigarrow P(\mathbf{x}^\mu) = \{x_1, \ldots, x_k\}$. Define $\mathrm{cls}(\mathbf{x}^\mu) = \min\{k \mid \mu_k > 0\}$.

Remark

Not every monomial ideal $\mathcal{I}$ has a finite Pommaret basis. A monomial ideal with finite Pommaret basis is called quasi-stable.

Example: Pommaret Basis

Example

- Consider $\mathcal{I} = \langle x^2, y^2 \rangle \trianglelefteq \mathbb{K}[x, y]$, we have $\mathrm{cls}(x^2) = 1$, $\mathrm{cls}(y^2) = 2$.
- We obtain a Pommaret basis by adding the generator x^2y.

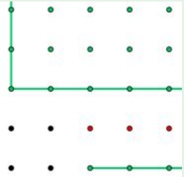


Figure: The ideal $\mathcal{I}$ and the Pommaret cones of its generators

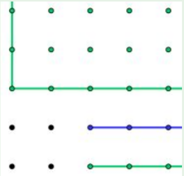


Figure: The Pommaret cone of x^2y added.

Pommaret-like Division

Definition (Hashemi et al. 2023)

The Pommaret-like division assigns $\mathrm{NMP}_P(u, U)$ as follows:

- all Janet-like non-multiplicative powers $x_a^{p_a}$ with $a > \mathrm{cls}(u)$.
- the Janet multiplicative variables x_b with $b > \mathrm{cls}(u)$.

Theorem (Hashemi et al., 2023)

A monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ has a finite Pommaret-like basis if and only if it is quasi-stable.

Pommaret-like basis for general ideal exists iff the leading ideal is quasi-stable.

Overview

	in $\mathcal{R}$	in $\mathcal{R}/\mathcal{I}$
Gröbner bases	Buchberger (1965)	e.g. Spear (1977), La Scala & Stillman (1998), Hashemi et al. (2021)
Involutive Bases	Janet (1920), Gerdt & Blinkov (1998), Seiler (2009)	Hashemi et al. (2021)
Involutive-like bases	Gerdt & Blinkov (2005) (Janet-like), Hashemi et al. (2023) (Pommaret-like)	Hashemi et al. (2025)

Term-ordering free analog of (relative) Pommaret bases (marked bases): see Cioffi & Roggero (2011), Ceria et al. (2015), Bertone et al. (2025).

Table of Contents

Standard grading, modules, and maps

- We equip the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ with the *standard grading*, for which $\deg(x_i) = 1$ for $1 \leq i \leq n$. Then

$$R = \bigoplus_{d \geq 0} R_d \,,$$

where R_d is the $\mathbb{K}$-vector space generated by the degree d homogeneous polynomials of R.

- A *graded R-module M* is an R-module M such that

$$M = \bigoplus_{d \in \mathbb{Z}} M_d \quad \text{and} \quad R_i \cdot M_d \subseteq M_{d+i} \,.$$

- Then each M_d is a $\mathbb{K}$-vector space as $\mathbb{K} \cdot M_d = R_0 \cdot M_d \subseteq M_d$.
- An R-linear map $\varphi : M \to N$ is *a-homogeneous* or *is graded of degree a* if $\varphi(M_d) \subseteq N_{d+a}$ for all $d \geq 0$.

Presentations and syzygies

- A *shifted copy of* R is a graded free module $R(-a)$, with $R(-a)_d = R_{d-a}$.
- We have $R(-a)_d = \{0\}$ for $d < a$, and $1 \in R(-a)_a$.

Theorem

Let M be a finitely generated graded R-module generated by elements $m_1, \ldots, m_k$ with degrees $a_1, \ldots, a_k$. Then $M \cong F/N$, where N is the graded submodule of $F = \oplus_{i=1}^{k} R(-a_i)$ given by

$$N = \ker(\pi) \quad \textit{for} \quad \pi : F \to M,\ \mathbf{e}_i \mapsto m_i\,.$$

Definition

With the notation of the theorem above, write $\mathbf{m} = (m_1, \ldots, m_k)$. Then N is called the *(graded) syzygy module of* $\mathbf{m}$, *written* $N = \mathrm{Syz}(\mathbf{m})$.

Module monomial orderings

- Write $\mathbf{e}_i$ for the ith standard basis vector of $F = \oplus_{i=1}^{k} R(-a_i)$.
- A *module monomial in F* is an element $x_1^{\mu_1} \cdots x_n^{\mu_n} \mathbf{e}_i =: x^\mu \mathbf{e}_i$ for some $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n$ and $i \in \{1, \ldots, k\}$.
- A total ordering $\prec$ of the monomials of F is a *module monomial ordering* if
  1. $\forall x^\mu \in \mathrm{Mon}(R) \setminus \{1\}$, $\mathbf{f} \in \mathrm{Mon}(F)$: $\mathbf{f} \prec x^\mu \mathbf{f}$,
  2. $\forall x^\mu \in \mathrm{Mon}(R)$, $\mathbf{f}, \mathbf{g} \in \mathrm{Mon}(F)$: $\mathbf{f} \prec \mathbf{g} \Longrightarrow x^\mu \mathbf{f} \prec x^\mu \mathbf{g}$.
- For $\mathbf{f} \in F$, define *leading monomial* $\mathrm{lm}_\prec(\mathbf{f})$, *leading coefficient* $\mathrm{lc}_\prec(\mathbf{f})$, *leading term* $\mathrm{lt}_\prec(\mathbf{f})$ w.r.t. $\prec$ accordingly.

Example (Schreyer)

Let $\prec$ be a monomial ordering on F and $H = \{\mathbf{h}_1, \ldots, \mathbf{h}_\ell\} \subset F$ an enumerated finite set of homogeneous elements of degrees $b_1, \ldots, b_\ell$. Then we obtain a monomial ordering $\prec_H$ on $\oplus_{i=1}^{\ell} R(-b_i)$ via

$$\begin{aligned} x^\mu \mathbf{e}_i \prec_H x^\nu \mathbf{e}_j :\Longleftrightarrow & [\mathrm{lm}(x^\mu \mathbf{h}_i) \prec \mathrm{lm}(x^\nu \mathbf{h}_j)] \\ & \vee [\mathrm{lm}(x^\mu \mathbf{h}_i) = \mathrm{lm}(x^\nu \mathbf{h}_j) \ \wedge \ j < i] \, . \end{aligned}$$

Module Gröbner bases

Definition

Let M be a finitely generated graded submodule of $F = \oplus_{i=1}^{k} R(-a_i)$ and let $\prec$ be a module monomial ordering on F. Then a finite generating set G of M is a *Gröbner basis* of M w.r.t $\prec$ if

$$\langle \mathrm{lm}_{\prec}(\mathbf{g}) \mid \mathbf{g} \in G \rangle = \langle \mathrm{lm}_{\prec}(\mathbf{m}) \mid \mathbf{m} \in M \setminus \{\mathbf{0}\} \rangle \,.$$

Theorem

Every finitely generated submodule of F has a unique reduced Gröbner basis for every module monomial ordering.
Module Gröbner bases can be computed by Buchberger's algorithm, where the S-Polynomial of $\mathbf{f}, \mathbf{g} \in F$ with $\mathrm{lm}(\mathbf{f}) = x^{\mu}\mathbf{e}_i$ and $\mathrm{lm}(\mathbf{g}) = x^{\nu}\mathbf{e}_j$ is

$$S(\mathbf{f}, \mathbf{g}) := \begin{cases} \frac{\mathsf{lcm}(x^{\mu}, x^{\nu})}{\mathrm{lc}(\mathbf{f}) \cdot x^{\mu}} \mathbf{f} - \frac{\mathsf{lcm}(x^{\mu}, x^{\nu})}{\mathrm{lc}(\mathbf{g}) \cdot x^{\nu}} \mathbf{g}, & i = j \\ \mathbf{0}, & \text{otherwise} \,. \end{cases}$$

Schreyer's construction (1)

- Let $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_\ell\} \subseteq F = \oplus_{i=1}^k R(-a_i)$ be the reduced Gröbner basis of $\langle G \rangle$ w.r.t. $\prec$, with degrees $b_1, \ldots, b_\ell$.
- Write $\mathrm{lt}(\mathbf{g}_i) = t_i \mathbf{e}_{\lambda(i)}$ for $1 \leq i \leq \ell$ and $t_{ij} := \mathrm{lcm}(t_i, t_j)$ if $i \neq j$ and $\lambda(i) = \lambda(j)$.
- For $1 \leq i < j \leq \ell$ with $\lambda(i) = \lambda(j)$ consider the relation in F obtained by the division algorithm applied to the S-Polynomial $S(\mathbf{g}_i, \mathbf{g}_j)$,

$$\frac{t_{ij}}{t_i}\mathbf{g}_i - \frac{t_{ij}}{t_j}\mathbf{g}_j = \sum_{k=1}^{\ell} h_{ijk}\mathbf{g}_k \,.$$

- Define

$$\mathbf{S}_{ij} := \frac{t_{ij}}{t_i}\mathbf{e}_i - \frac{t_{ij}}{t_j}\mathbf{e}_j - \sum_{k=1}^{\ell} h_{ijk}\mathbf{e}_k \in \bigoplus_{i=1}^{\ell} R(-b_i)\,.$$

Schreyer's construction (2)

Lemma

$$\begin{aligned}&\mathrm{Syz}(\mathrm{lm}(\mathbf{g}_1), \ldots, \mathrm{lm}(\mathbf{g}_\ell)) \\ &= \left\langle \frac{t_{ij}}{t_i}\mathbf{e}_i - \frac{t_{ij}}{t_j}\mathbf{e}_j \mid 1 \le i < j \le \ell, \lambda(i) = \lambda(j) \right\rangle \subseteq \bigoplus_{i=1}^{\ell} R(-b_i)\,.\end{aligned}$$

Theorem (Schreyer 1980)

1. $\mathrm{lm}_{\prec_G}(\mathbf{S}_{ij}) = \frac{t_{ij}}{t_i}\mathbf{e}_i$
2. $\{\mathbf{S}_{ij} \mid 1 \le i < j \le \ell,\ \lambda(i) = \lambda(j)\}$ *is a Gröbner basis of* $\mathrm{Syz}(G)$ *w.r.t.* $\prec_G$.

Free Resolutions

Definition

Free resolution **F** of finitely generated graded R-module M:
Finitely generated free R-modules $F_0, F_1, \ldots$ of ranks $r_0, r_1, \ldots$;
0-homogeneous R-maps $\delta_0, \delta_1, \ldots$:

$$\mathbf{F}: \quad 0 \xrightarrow{0} F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \to 0,$$

with $\mathrm{im}(\delta_0) = M$ and $\mathrm{im}(\delta_{m+1}) = \ker(\delta_m)$ for all m.
Differential of **F**: Collection $\{\delta_0, \ldots, \delta_n\}$.

Remark

δ_m is represented by matrix D_m with entries in R, and $D_m \cdot D_{m+1} = 0$.

Application of Syzygies

The set G_m of columns of D_m is, for $m \geq 1$, a homogeneous generating set of the iterated syzygy module $\mathrm{Syz}^m(G)$, where $G := \{\delta_0(\mathbf{e}_1), \ldots, \delta_0(\mathbf{e}_{r_0})\}$.

Minimal free resolutions

Definition

Resolution $\mathbf{F}$ is *minimal*: $\Longleftrightarrow$ no constant terms $\neq 0$ in matrices D_m.

Definition

Free resolutions $(\mathbf{F}, \delta)$ and $(\mathbf{G}, \partial)$ of M *isomorphic* if there are 0-homog. isomorphisms $\alpha_i : F_i \to G_i$ with $\alpha_{i-1} \circ \delta_i = \partial_i \circ \alpha_i$, where $\alpha_{-1} := \mathrm{id}_M$.

Lemma (Graded Nakayama Lemma)

Let M be a finitely generated graded R-module, $U \subseteq M$ a graded submodule and $J \subsetneq R$ a proper homogeneous ideal. Then

1. *If $M = JM$, then $M = \{0\}$.*
2. *If $M = U + JM$, then $U = M$.*

Theorem

The minimal free resolution of M is unique up to isomorphism.

Infinite Free Resolutions

Remark

Free resolution **F** *of homogeneous* $\mathcal{J} \trianglelefteq \mathcal{R}/\mathcal{I}$*: needs infinitely many* $\mathcal{R}/\mathcal{I}$*-modules* $F_0, F_1, \ldots$ *and homogeneous* $\mathcal{R}/\mathcal{I}$*-linear maps* $\delta_0, \delta_1, \ldots$*:*

$$\mathbf{F}: \quad \cdots \xrightarrow{\delta_{m+2}} F_{m+1} \xrightarrow{\delta_{m+1}} F_m \xrightarrow{\delta_m} F_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} \mathcal{J} \to 0,$$

with $\mathrm{im}(\delta_0) = \mathcal{J}$ *and* $\mathrm{im}(\delta_{m+1}) = \ker(\delta_m)$ *for all* $m \geq 0$*.*

Minimalization of a free resolution

Given a non-minimal finite free resolution $\mathbf{F}$ of M, there is a differential δ_m whose matrix D_m contains a constant. We can minimalize $\mathbf{F}$ by iteratively proceeding as follows (e.g. Cox et al. 2005 (textbook)):

1. Pick entry $d_{ij} \in \mathbb{K}^\times$ in D_m and add suitable multiples of the jth column to the other columns to obtain D'_m with d_{ij} as only non-zero entry in ith row;
2. Remove the jth free summand of F_m and the ith free summand of F_{m-1};
3. Delete the ith row of D_{m+1}, the ith row and jth column of D'_m and the ith column of D_{m-1};
4. Leave all other maps and free modules unchanged.

The result is another free resolution of M of lower overall rank. If the input is a finite free resolution, the process stops after finitely many steps with a minimal free resolution of M.

Betti numbers and Betti diagram

- Let $(\mathbf{F}, \delta)$ be a minimal free resolution of M. Write

$$F_i = \bigoplus_{j \geq 0} R(-j)^{b_{ij}} .$$

- As the 0-homogeneous isomorphisms between minimal free resolutions preserve ranks and degree shifts, the numbers $b_{ij} = b_{ij}(M)$ are invariants of M, called *graded Betti numbers*.
- The *total Betti numbers* are $b_i(M) := \sum_{j \geq 0} b_{ij}(M)$.
- The *Betti diagram* is

$$\begin{array}{c|cccc} & b_0 & b_1 & b_2 & \dots \\ \hline 0 & b_{0,0} & b_{1,1} & b_{2,2} & \dots \\ 1 & b_{0,1} & b_{1,2} & b_{2,3} & \dots \\ 2 & b_{0,2} & b_{1,3} & b_{2,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Poincaré series, Hilbert series, and other invariants (1)

We look at some invariants of M and at how they relate to each other. (Cf. Peeva 2011 (textbook))

- *Projective dimension*: $\mathrm{pdim}(M) = \max\{i \mid b_i(M) \neq 0\}$.
- *Castelnuovo–Mumford regularity*:
  $\mathrm{reg}(M) = \max\{j \mid \exists i :\ b_{i,i+j}(M) \neq 0\}$.
- *Poincaré series*: $P_M(t) = \sum_{i \geq 0} b_i(M) t^i$
- *Graded Poincaré series*: $\mathrm{P}_M(t, z) = \sum_{i \geq 0} \sum_{j \geq 0} b_{ij}(M) t^i z^j$.
- *Hilbert series*:

$$H_M(z) = \frac{\sum_{i \geq 0} \sum_{j \geq 0} (-1)^i b_{ij}(M) z^j}{(1-z)^n} =: \frac{E_M(z)}{(1-z)^n}$$

- *Hilbert polynomial*: Unique polynomial $\mathrm{HP}_M \in \mathbb{Q}[z]$ of degree less than n such that $\mathrm{HP}_M(j) = \dim_{\mathbb{K}}(M_j)$ for $j \gg 0$.

Poincaré series, Hilbert series, and other invariants (2)

We have (Cf. Peeva 2011 (textbook))

- $P_M(t) = \mathrm{P}_M(t, 1)$;
- $H_M(z) = \mathrm{P}_M(-1, z)/(1-z)^n$;
- $\deg(\mathrm{HP}_M) = \dim(M) - 1$;
- if q is the maximal exponent such that $(1-z)^q$ divides $E_M(z)$, then for $h_M(z) := E_M(z)/(1-z)^q$, $h_M(1)$ is the *multiplicity* of M;
- $H_M(z) = h_M(z)/(1-z)^{\dim(M)}$;
- $\mathrm{lc}(\mathrm{HP}_M) = h_M(1)/(\dim(M) - 1)!$;
- if $h_M(z) = h_r z^r + \cdots h_1 z + h_0$, then

$$\mathrm{HP}_M(i) = \sum_{j=0}^{r} h_j \binom{\dim(M) - 1 + i - j}{\dim(M) - 1};$$

- $\dim_{\mathbb{K}}(M_i) = \mathrm{HP}_M(i)$ for $i \geq \deg(h_M)$.

Table of Contents

Relative Involutive-like Bases

We work in the quotient ring $\mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is a homogeneous ideal.

Definition (Hashemi et al. 2021)

$H = \{h_1, \dots, h_k\} \subset \mathcal{J}$ Gröbner basis of $\mathcal{J} \supset \mathcal{I}$ relative to $\mathcal{I}$ if

$$\langle \mathrm{lt}(H) \rangle + \mathrm{lt}(\mathcal{I}) = \mathrm{lt}(\mathcal{J}).$$

Remark (Hashemi et al. 2023+2025)

- *Concept of involutive-like divisions and bases carries over to the quotient ring setting: Obtain division $L_{\mathcal{I}}$ relative to $\mathcal{I}$*
- *Thus, we have the concepts of relative Janet(-like) and Pommaret(-like) bases.*
- *Pommaret(-like) bases exist in relative quasi-stable position.*

Relative Quasi-Stability and Quasi-Stable Position

Definition (Hashemi et al. 2021)

Let $\mathcal{I} \subseteq \mathcal{J}$ be two monomial ideals. $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ if for all $\mathbf{x}^\mu \in \mathcal{J} \setminus \mathcal{I}$ and $x_i \notin P(\mathbf{x}^\mu)$ there exists an $s \geq 0$ such that either $x_i^s \cdot \mathbf{x}^\mu \in \mathcal{I}$ or $x_i^s \cdot \mathbf{x}^\mu / x_{\mathrm{cls}(x^\mu)} \in \mathcal{J}$.

Proposition (Hashemi et al. 2021+2023+2025)

Let $\mathcal{I} \subseteq \mathcal{J}$ be two monomial ideals. Then $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ $\iff$ $\mathcal{J}$ has a finite Pommaret-like basis relative to $\mathcal{I}$.

- Special setting: $\mathcal{I} \subseteq \mathcal{J}$ *homogeneous* ideals, $\prec$ degree reverse lexicographical ordering with $x_1 \prec x_2 \prec \cdots \prec x_n$.
- Can reach quasi-stable positions for $\mathcal{I}$ and for $\mathcal{J}$ relative to $\mathcal{I}$ by concatenating coordinate transformations of the form $x_i \mapsto x_i + x_j$, where $i < j$ and $x_k \mapsto x_k$ for $k \neq i$.
- In these coordinates, have Pommaret-like bases both for $\mathcal{I}$ and for $\mathcal{J}$ relative to $\mathcal{I}$.

Relative Schreyer Theorem

Given: Ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ and $F \subset \mathcal{R} \setminus \mathcal{I}$.

- Can produce new leading terms by:
  1. S-polynomials among the elements of F
  2. annihilating leading terms modulo $\mathrm{lt}(\mathcal{I}) \rightsquigarrow$ A-polynomial
- S-polynomials and A-polynomials $\rightsquigarrow$ S-syzygies and A-syzygies

Theorem (Hashemi et al., 2021)

H Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I} \Longrightarrow$
$\{S\text{−}syzygies\} \cup \{A\text{−}syzygies\}$ Gröbner basis of $\mathrm{Syz}_{\mathcal{R}/\mathcal{I}}([h] \mid h \in H)$

Remark

The induced free resolution over $\mathcal{R}/\mathcal{I}$ is, in general, of infinite length.

A first example: relative Pommaret bases

Example

$\mathcal{R} = \mathbb{K}[x, y, z]$, $\mathcal{I} = \langle z^3 \rangle$, $\mathcal{J} = \langle xyz, y^2z, yz^2, \mathcal{I} \rangle$.

$$\cdots \to (\mathcal{R}/\mathcal{I})^4 \overset{\varphi_4}{\to} (\mathcal{R}/\mathcal{I})^4 \overset{\varphi_3}{\to} (\mathcal{R}/\mathcal{I})^4 \overset{\varphi_2}{\to} (\mathcal{R}/\mathcal{I})^4 \overset{\varphi_1}{\to} (\mathcal{R}/\mathcal{I})^3 \overset{\varphi_0}{\to} \mathcal{J}/\mathcal{I} \to 0$$

Mappings φ_k given by matrices D_k, with:

$$D_0 = \begin{pmatrix} xyz & y^2z & yz^2 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},$$

$$D_2 = \begin{pmatrix} z & 0 & 0 & 0 \\ -y & z^2 & 0 & 0 \\ x & 0 & z^2 & 0 \\ 0 & xz & yz & z^2 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} z^2 & 0 & 0 & 0 \\ y & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -x & -y & z \end{pmatrix},$$

and $D_4 = D_2 \quad \rightsquigarrow \quad$ periodic resolution!

Free resolutions from relative Pommaret-like bases

Definition

$\mathcal{I} = \langle x_i^{a_i}, x_{i+1}^{a_{i+1}}, \ldots, x_n^{a_n} \rangle \trianglelefteq \mathcal{R}$ with $2 \leq a_n \leq a_{n-1} \leq \cdots \leq a_{i+1} \leq a_i$ is *Clements-Lindström (C-L)* monomial ideal. $\mathcal{R}/\mathcal{I}$ is a C-L ring.

Theorem (Hashemi et al. 2025)

Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$, H minimal Pommaret-like basis of $\mathcal{J}$ relative to $\mathcal{I}$. For each $h \in H$ take:

1. *Syzygies $\mathbf{S}_{h,x_i}$ ($x_i^{p_i} \in \mathrm{NMP}_{P_\mathcal{I}}(\mathrm{lt}(h), \mathrm{lt}(H))$) induced by non-multiplicative powers,*
2. *Syzygies $\mathbf{S}_{h,\mathbf{x}^\mu}$, where $\mathbf{x}^\mu$ is a $P_\mathcal{I}$-multiplier, induced by annihilating $\mathrm{lt}(h)$ mod $\mathrm{lt}(\mathcal{I})$.*

Then $H_1 = \{\mathbf{S}_{h,x_i} \mid h \in H\} \cup \{\mathbf{S}_{h,\mathbf{x}^\mu} \mid h \in H\}$ is a minimal Pommaret-like basis of $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$ for a suitable module term ordering.

Iteration $\rightsquigarrow$ infinite induced $\mathcal{R}/\mathcal{I}$-free resolution.

Minimal Pommaret-like Resolutions

Theorem (Hashemi et al. 2025)

$\mathcal{I} \trianglelefteq \mathcal{R}$ C-L ideal, $\mathcal{J} \supset \mathcal{I}$ monomial ideal generated by minimal relative Pommaret-like basis $H \subset (\mathcal{J} \setminus \mathcal{I}) \cap \mathcal{T}$. If H is the minimal generating set of $\mathcal{J}$ relative to $\mathcal{I}$ and for each $t \in H$ and $x_a^{p_a} \in \mathrm{NMP}_{P_\mathcal{I}}(t, H)$, the unique $P_\mathcal{I}$-divisor $s \in H$ of $t \cdot x_a^{p_a}$ fulfils $\mathrm{cls}(s) > \mathrm{cls}(t)$, then the free resolution of $\mathcal{J}$ over $\mathcal{R}/\mathcal{I}$ induced by H is minimal.

Remark

We obtain bases for the free modules in the resolution of $\mathcal{J} = \langle \mathcal{I}, H \rangle$:

1. *F_0 has basis $\{\mathbf{e}_\alpha \mid h_\alpha \in H\}$.*
2. *For $r \geq 1$, F_r has basis elements $\mathbf{e}_{\alpha,\mathbf{x}^\mu}$, where $\deg(\mathbf{x}^\mu) = r$ and $\mathrm{cls}(\mathbf{x}^\mu) \geq \mathrm{cls}(t_\alpha)$. Moreover, x^μ is squarefree in variables of index $< i$.*

Thus we obtain the Betti numbers and the rational Poincaré series of $\mathcal{J}$ in terms of binomial coefficient expressions.

Different Concepts of Minimality

- Resolution obtained is in general not minimal (differentials may contain constants)
- Generally: minimal free resolution smaller than resolution by reduced Gröbner bases smaller than resolution by minimal Pommaret bases
- But: Pommaret resolution highly structured, combinatorial decompositions of each syzygy module $\mathrm{Syz}^k_{\mathcal{P}/\mathcal{I}}(H)$

Upper bounds

For resolutions over CL-rings, we obtain a coefficient-wise upper bound for the Poincaré series in terms of binomial coefficient expressions.

Example resolution

$\mathcal{I} = \langle y^4, z^5 \rangle$ and $\mathcal{J} = \langle \mathcal{I}, H \rangle$, with minimal relative Pommaret-like basis $H = \{x^2y^3, xy^2z^2, y^3z^2, z^3\} \rightsquigarrow$ minimal free resolution of $\mathcal{J}/\mathcal{I}$ over $\mathcal{R}/\mathcal{I}$:

$$D_0 = \begin{pmatrix} x^2y^3 & xy^2z^2 & y^3z^2 & z^3 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} y & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 & 0 \\ 0 & -x^2 & -x & 0 & y & z & 0 \\ 0 & 0 & 0 & -xy^2 & 0 & -y^3 & z^2 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} y^3 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & z^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^3 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y & z^4 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & 0 & xy^2 & 0 & 0 & y^3 & z & 0 & 0 \\ 0 & 0 & x^2z^2 & 0 & x & 0 & 0 & -y & z^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy^2z^2 & 0 & 0 & y^3z^2 & z^3 \end{pmatrix}, \ldots$$

The total Betti numbers are: $\operatorname{rank}(F_r) = 3\binom{1+r}{1} + \binom{r}{0} = 4 + 3r$.

Related results

Our constructions contain as special cases:

- The Eliahou–Kervaire (1990) resolution of stable monomial ideals in $\mathcal{R}$,
- The Pommaret–Seiler (Seiler 2009) resolution of quasi-stable ideals in $\mathcal{R}$,
- The Gasharov–Murai–Peeva (2011) resolution of squarefree Borel ideals in zero-dimensional C-L rings.

Table of Contents

Motivation

Definition

We study $I_{n,\mathbf{m},k} = (x_1^{m_1}, \dots, x_n^{m_n}, (x_1 + \cdots + x_n)^k) \subset \mathbf{k}[x_1, \dots, x_n]$, where $\mathrm{char}(\mathbf{k}) = 0$, $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geq 2}$ and $k \geq 1$.

- These ideals and their algebras can be regarded as generic because they attain the Hilbert series of Fröberg's conjecture (Stanley 1980; Watanabe 1989).
- The algebras are *almost complete intersections* and have attracted much attention recently.
  - Booth, Singh, and Vraciu 2024
  - Jonsson Kling, Lundqvist, Mohammadi, O., Sáenz-de-Cabezón, 2024
  - Diethorn, Güntürkün, Hardesty, Mete, Şega, Sobieska, 2025
  - Jonsson Kling, Lundqvist, Mohammadi, O., 2025, to appear in Transactions of the AMS
- Our approach is the first to obtain explicit Gröbner bases and yields new, independent proof of Stanley's and Watanabe's results.

Example and methods for squarefree algebra case

Write $I_{n,(2,\dots,2),k} = I_{n,k}$.

Example

The reduced Gröbner basis of $I_{5,2}$ is

$$G_{5,2} = \{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, \\ g_{\{1,2\},5,2}, g_{\{1,3,4\},5,2}, g_{\{2,3,4\},5,2}\},$$

where

$$\begin{aligned} g_{\{1,2\},5,2} &= x_1x_2 + x_1x_3 + \cdots + x_4x_5, \\ g_{\{1,3,4\},5,2} &= x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_3x_4x_5, \\ g_{\{2,3,4\},5,2} &= x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 . \end{aligned}$$

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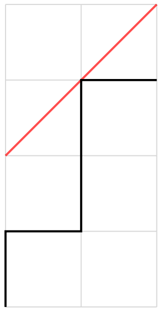
Associate lattice paths to leading monomials.
$\mathrm{in}(g_{\{1,3,4\},5,2}) = x_1x_3x_4$ is shown on the right.



Results for squarefree algebra

Theorem ([JLMOS 2024])

Consider the family $\mathcal{A}$ of subsets $A \subseteq [n]$ satisfying $\max(A) = 2|A| - k$ *for some* $k \geq 2$*, and minimal with respect to inclusion.*

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$$G_{n,k} = \{x_1^2, \ldots, x_n^2\} \cup \bigcup_{d=k}^{k+\lfloor (n-k)/2 \rfloor} \{g_{A,n,k} \mid A \in \mathcal{A}, |A| = d\},$$

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where for $|A| = d$,

$$g_{A,n,k} = e_d(x_{i_1}, \ldots, x_{i_{n-d+k}})$$

is the elementary symmetric polynomial of degree d in the variables indexed by the set $\{i_1, \ldots, i_{n-d+k}\} = A \cup \{2d - k + 1, \ldots, n\}$ with leading term $x_A = \prod_{a \in A} x_a$.

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New phenomena in non-squarefree case

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- More than just 0-1-Coefficients $\rightsquigarrow$ need coefficient rules.
- "Suffixes" of GB leading monomials show more variety than $x_{n-1}x_n$ $\rightsquigarrow$ need other lattice path criterion than just touching some line to describe initial ideal.

Properties of Artinian monomial complete intersections

Monomial ideals $P_{n,\mathbf{m}} = (x_1^{m_1}, \ldots, x_n^{m_n}) \in \mathbb{Z}^n_{\geq 2}$ define Artinian complete intersection algebras $A_{n,\mathbf{m}} = R/P_{n,\mathbf{m}}$ with the following properties

(i) $A_{n,\mathbf{m}}$ has the strong Lefschetz property.

(ii) The socle degree $D_{n,\mathbf{m}}$ of $A_{n,\mathbf{m}}$ is given by $D_{n,\mathbf{m}} = \sum_{i=1}^{n}(m_i - 1)$.

(iii) The Hilbert series of $A_{n,\mathbf{m}}$ is symmetric.

$$[t^d]\,\mathrm{HS}(A_{n,\mathbf{m}}; t) = [t^{D_{n,\mathbf{m}}-d}]\,\mathrm{HS}(A_{n,\mathbf{m}}; t)\,.$$

(iv) The Hilbert series of $A_{n,\mathbf{m}}$ is unimodal.

These are essentially C–L rings without requirement of ordered exponents.
The symmetry of the Hilbert series is very important for our work.

Lattice paths

Definition

A monomial $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in R$ is called $\mathbf{m}$-*free* if $\alpha_i < m_i$ for $1 \leq i \leq n$.

From an $\mathbf{m}$-free monomial, we obtain an $\mathbf{m}$-admissible lattice path

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- $x_i^0 \rightsquigarrow (i-1, b) \to (i, b+1) \quad$ (1-up),

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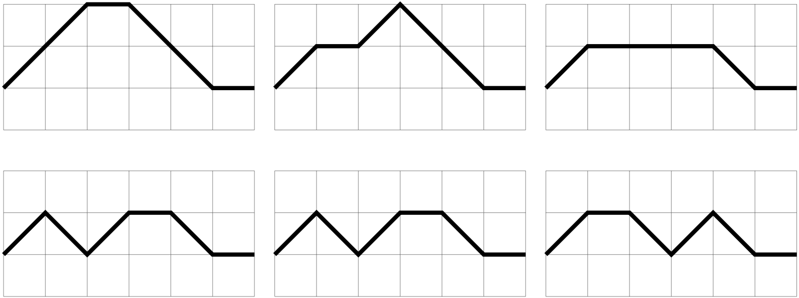
- $x_i^e \rightsquigarrow (i-1, b) \to (i, b-e+1) \quad$ ($(e-1)$-down).

Lemma ([JLMO 2025])

Fix positive integers n, d and a vector $\mathbf{m} \in \mathbb{Z}^n_{\geq 2}$. Then the number of $\mathbf{m}$-admissible lattice paths ending at $(n, n-d)$ equals $\dim_{\mathbf{k}}(R/P_{n,\mathbf{m}})_d$.

Observation on Motzkin paths

In the case $\mathbf{m} = (3, \ldots, 3)$ and $k = 1$, associating lattice paths to GB leading monomials, observe that these paths are similar to Motzkin paths.



Number scheme and reflection points

For $1 \leq j \leq n$, let $\mathbf{m}_j = (m_1, \ldots, m_j)$. We arrange the coefficients of $\mathrm{HS}(\mathbf{k}; t),\ \mathrm{HS}(\mathbf{k}[x_1]/P_{1,\mathbf{m}_1}; t), \ldots$ as columns in a number scheme.

$$
\begin{array}{ccccc}
 & & & & 1 \\
 & & & 1 & 4 \\
 & & 1 & 3 & 8 \\
 & 1 & 2 & 4 & 10 \\
1 & 1 & 2 & 3 & 8 \\
 & 1 & 1 & 1 & 4 \\
 & & & & 1
\end{array}
$$

Figure: Here, $\mathbf{m} = (3, 2, 2, 3)$, and $k = 2$. A piecewise linear function interpolates between the points located $k/2$ below the midpoints of the columns.

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Figure: Here, $\mathbf{m} = (3, 2, 2, 3)$, and $k = 2$. A piecewise linear function interpolates between the points located $k/2$ below the midpoints of the columns.

Example

Let $\mathbf{m} = (3, 2, 2, 3)$ and $t = x_1 x_3 x_4^2$. Its path ends at the point $(4, 0) = (4, n - \deg(t))$, where it touches the red line.

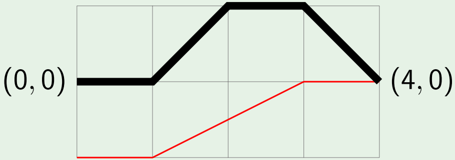

Figure: The graph of $L_{4,(3,2,2,3),2}$ and the lattice path associated with the monomial $x_1 x_3 x_4^2$.

Reflection and critical paths

Definition ([JLMO 2025])

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$$ \begin{aligned} \Lambda : \{\mathbf{P} \mid \mathbf{P} \text{ critical and } \mathbf{m}\text{-admissible}\} &\to \{\mathbf{P} \mid \mathbf{P}\ \mathbf{m}\text{-admissible}\}, \\ \mathbf{P} &\mapsto {}^{\lambda(\mathbf{P})}\mathbf{P}'. \end{aligned} $$

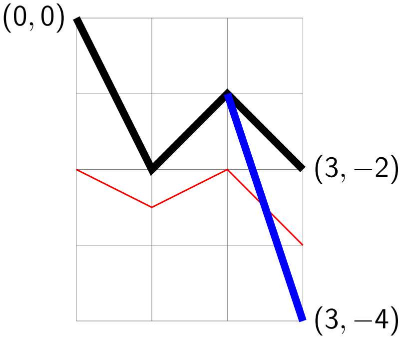
Critical path ideal

Definition ([JLMO 2025])

Let $M_{n,\mathbf{m},k}$ be the set of monomials corresponding to critical $\mathbf{m}$-admissible lattice paths. We write $(M_{n,\mathbf{m},k})$ for the ideal generated by $M_{n,\mathbf{m},k} + P_{n,\mathbf{m}}$.

Example

Consider the lattice path corresponding to the $(4,2,5)$-free monomial $x_1^3 x_3^2$, shown in black. Its image under the reflection map Λ is such that the initial two segments remain in black and the reflected portion appears in blue.



Central counting argument and Hilbert series

Definition ([JLMO 2025])

Let d be the degree of an $\mathbf{m}$-free monomial and $d' = (\sum m_i) - n - d + k$. We define the degree-wise reflection map Λ_d as the restriction of Λ to the set of critical $\mathbf{m}$-admissible paths ending at $(n, n-d)$

$$\Lambda_d : \left\{ \begin{array}{l} \mathbf{P} \text{ critical with} \\ \text{endpoint } (n, n-d) \end{array} \right\} \to \left\{ \begin{array}{l} \mathbf{P} \text{ critical with} \\ \text{endpoint } (n, n-d') \end{array} \right\}, \quad \mathbf{P} \mapsto \Lambda(\mathbf{P})\,.$$

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Proposition ([JLMO 2025])

(i) *Λ_d is invertible with $\Lambda_d^{-1} = \Lambda_{d'}$.*

(ii) *The Hilbert series of $R/(M_{n,\mathbf{m},k})$ and $R/P_{n,\mathbf{m}}$ satisfy*

$$\mathrm{HS}\left(R/(M_{n,\mathbf{m},k}); t\right) = \left[(1 - t^k)\, \mathrm{HS}(R/P_{n,\mathbf{m}}; t)\right],$$

where the brackets denote truncation at the first coefficient ≥ 0.

Minimal generators

The set of minimal critical paths,

$$\mathrm{Crit}_{n,\mathbf{m},k} = \bigcup_{j=1}^{n} \mathrm{Crit}_{n,\mathbf{m},k,j}\,,$$

can be obtained by inspection of reflections. It is partitioned into the subsets $\mathrm{Crit}_{n,\mathbf{m},k,j}$ of generators having x_j as variable of highest index.

Theorem ([JLMO 2025])

The ideal $(M_{n,\mathbf{m},k})$ is minimally generated by the set

$$S_{n,\mathbf{m},k} = \{x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n}\} \cup \mathrm{Crit}_{n,\mathbf{m},k}\,.$$

where $x_j^{m_j}$ is removed if $k + \sum_{i=1}^{j-1}(m_i - 1) < m_j$.

The Gröbner basis

We explicitly construct polynomials g_s with $\mathrm{in}(g_s) = s$ for every minimal generator s of $(M_{n,\mathbf{m},k})$. We obtain:

Theorem ([JLMO 2025])

The reduced Gröbner basis of $I_{n,\mathbf{m},k}$ for $x_1 \succ \cdots \succ x_n$ is

$$G_{n,\mathbf{m},k} = \{x_1^{m_1}, \ldots, x_n^{m_n}\} \cup \{g_s \mid s \in \mathrm{Crit}_{n,\mathbf{m},k}\},$$

without $x_j^{m_j}$ if $k + \sum_{i=1}^{j-1}(m_i - 1) < m_j$. For $s = s' x_j^{\deg_{x_j}(s)} \in \mathrm{Crit}_{n,\mathbf{m},k,j}$,

$$g_s = \sum_{s'' | s'} \lambda_{s''} \, s''(x_j + \cdots + x_n)^{\deg(s) - \deg(s'')} \mod P_{n,\mathbf{m}},$$

where

$$\lambda_{s''} = \left(\prod_{i=1}^{n-1} \frac{s_i!}{s_i''!} \binom{m_i - s_i'' - 1}{s_i - s_i''} \right) \cdot \frac{s_n!}{(\deg(s) - \deg(s''))!}.$$

Proof idea

The following lemma is shown via an application of inclusion-exclusion.

Lemma ([JLMO 2025])

Let p, q, and r be monomials such that $pq = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $r | pq$. Then

$$\sum_{v | \gcd(p,r)} (-1)^{\deg(v)} \left(\prod_{i=1}^{n} \binom{r_i}{v_i} \binom{\alpha_i - v_i}{p_i - v_i} \right) = \prod_{i=1}^{n} \binom{\alpha_i - r_i}{p_i}.$$

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- $\mathrm{HS}(R/I_{n,\mathbf{m},k};t) = \left[(1-t^k)\,\mathrm{HS}(R/P_{n,\mathbf{m}};t)\right]$ follows.

Generalized Catalan numbers

The *s-binomial coefficients* $\binom{n}{d}_s$ are defined for $n, s \in \mathbb{N}$ via

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Definition (e.g. [Ghemit and Ahmia 2024, Linz 2022])

The coefficients of the *s-Catalan triangle* are given by

$$C^{(s)}_{0,0} = 1 \quad \text{and} \quad C^{(s)}_{n,k} = \binom{2n}{sn+k}_s - \binom{2n}{sn+k+1}_s$$

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for $n > 0$ and $k \geq 0$. The s-Catalan triangle, written with entry $C^{(s)}_{n,k}$ in row n and column k, contains the s-Catalan numbers in its first column.

Log-concavity and log-convexity

s-Catalan numbers appear as structure coefficients that relate irreducible representations of the general linear group $\mathrm{GL}_n(\mathbf{k})$ (Linz 2022).

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Using our results, we provide the following lattice path interpretation of the numbers in the $(m-1)$-Catalan triangle for any positive integer $m-1$.

Proposition ([JLMO 2025])

For $m \geq 2$, $n \geq 1$, and $k \geq 0$, $C_{n,k}^{(m-1)}$ counts the m-admissible lattice paths of degree $n(m-1)-k$ whose $\mathbf{m}$-free monomials are not in $\mathrm{in}(I_{2n,\mathbf{m},1})$. The rows of the $(m-1)$-Catalan triangle are log-concave.

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Thank you